

ON PLIŠ METRIC ON THE SPACE OF STRICTLY CONVEX COMPACTA

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ABSTRACT. We consider a certain metric on the space of all convex compacta in \mathbb{R}^n , introduced by A. Pliš. The set of strictly convex compacta is a complete metric subspace of the metric space of convex compacta with respect to this metric. We present some applications of this metric to the problems of set-valued analysis, in particular we estimate the distance between two compact sets with respect to this metric and to the Hausdorff metric.

1. INTRODUCTION

We begin by some definitions for a finite-dimensional Euclidean space $(\mathbb{R}^n, \|\cdot\|)$ over \mathbb{R} with the inner product (\cdot, \cdot) . Let $B_r(a) = \{x \in \mathbb{R}^n \mid \|x - a\| \leq r\}$. Let $\text{cl } A$ denote the *closure* and $\text{int } A$ the *interior* of the subset $A \subset \mathbb{R}^n$. The *diameter* of the subset $A \subset \mathbb{R}^n$ is defined by $\text{diam } A = \sup_{x, y \in A} \|x - y\|$. The *distance* from the point $x \in \mathbb{R}^n$ to the set $A \subset \mathbb{R}^n$ is given by the formula $\varrho(x, A) = \inf_{a \in A} \|x - a\|$. We shall denote the *convex hull* of a set $A \subset \mathbb{R}^n$ by $\text{co } A$. We shall denote the *conic hull* of a set $A \subset \mathbb{R}^n$ by $\text{cone } A$ (cf. [1, 9, 13]).

The *Hausdorff distance* between two subsets $A, B \subset \mathbb{R}^n$ is defined as follows $h(A, B) =$

$$= \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\} = \inf \{r > 0 \mid A \subset B + B_r(0), B \subset A + B_r(0)\}.$$

The *supporting function* of the subset $A \subset \mathbb{R}^n$ is defined as follows

$$s(p, A) = \sup_{x \in A} (p, x), \quad \forall p \in \mathbb{R}^n. \quad (1.1)$$

The supporting function of any set A is always lower semicontinuous, positively uniform and convex. If the set A is bounded then the supporting function is Lipschitz continuous [1, 9].

It follows from the Separation Theorem (cf. [9, Lemma 1.11.4]) that for any convex compacta A, B in \mathbb{R}^n

$$h(A, B) = \sup_{\|p\|=1} |s(p, A) - s(p, B)|. \quad (1.2)$$

Let (T, ϱ) be a metric space. We say that a set-valued mapping $F : (T, \varrho) \rightarrow 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ is *upper semicontinuous* at the point $t = t_0$ if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall t : \varrho(t, t_0) < \delta \quad F(t) \subset F(t_0) + B_\varepsilon(0),$$

and *lower semicontinuous* at the point $t = t_0$ if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall t : \varrho(t, t_0) < \delta \quad F(t_0) \subset F(t) + B_\varepsilon(0).$$

We say that a set-valued mapping $F : (T, \varrho) \rightarrow 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ is *continuous* at the point $t = t_0$ if F is upper and lower semicontinuous at the point $t = t_0$.

We say that a set-valued mapping $F : (T, \varrho) \rightarrow 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ is *(upper, lower) (semi)continuous* on the set T , if F is (upper, lower) (semi)continuous at any point $t_0 \in T$.

Date: December 20, 2011.

2010 Mathematics Subject Classification. Primary: 54A20, 52A41. Secondary: 52A20, 52A99, 46N10.

Key words and phrases. Metric space, strictly convex compactum, modulus of convexity, set-valued mapping, strict convexity, uniform convexity, supporting function, Demyanov distance, Hausdorff distance.

For any convex compact set $A \subset \mathbb{R}^n$ and any vector $p \in \mathbb{R}^n$, the subset $A(p) = \{x \in A \mid (p, x) = s(p, A)\}$ is the subdifferential of the supporting function $s(p, A)$ at the point p . The set-valued mapping $\mathbb{R}^n \ni p \rightarrow A(p)$ is always upper semicontinuous (cf. [1, 13]).

A convex compactum in \mathbb{R}^n is called *strictly convex* if its boundary contains no nondegenerate line segments.

Definition 1.1. ([10]). Let E be a Banach space and let a subset $A \subset E$ be convex and closed. The *modulus of convexity* $\delta_A : [0, \text{diam } A] \rightarrow [0, +\infty)$ is the function defined by

$$\delta_A(\varepsilon) = \sup \left\{ \delta \geq 0 \mid B_\delta \left(\frac{x_1 + x_2}{2} \right) \subset A, \forall x_1, x_2 \in A : \|x_1 - x_2\| = \varepsilon \right\}.$$

Definition 1.2. ([10]). Let E be a Banach space and let a subset $A \subset E$ be convex and closed. If the modulus of convexity $\delta_A(\varepsilon)$ is strictly positive for all $\varepsilon \in (0, \text{diam } A)$, then we call the set A *uniformly convex* (with modulus $\delta_A(\cdot)$).

For any uniformly convex set A the modulus δ_A is a strictly increasing function on the segment $[0, \text{diam } A]$. In the finite-dimensional case the class of strictly convex compacta coincides with the class of uniformly convex compacta with moduli of convexity $\delta_A(\varepsilon) > 0$ for all permissible $\varepsilon > 0$ (cf. [3]).

We shall use $*$ for objects from conjugate space E^* : $\|\cdot\|_*$ is the norm in E^* , $B_1^*(0)$ is the unit closed ball in E^* and so on.

Following Plís [8] we define the metric ρ which is the main objective of the present paper.

Definition 1.3. ([8, Formula (3)]) The metric ρ on the space of convex compacta in \mathbb{R}^n is defined by the formula

$$\rho(A, B) = \sup_{\|p\|=1} h(A(p), B(p)), \quad (1.3)$$

for any convex compacta $A, B \subset \mathbb{R}^n$.

Definition 1.3 coincides with the definition of the *Demyanov metric* (see its definition in [4, Formula (4.1)]) – this was proved in [6].

The Hausdorff metric is the most natural metric for various questions of set-valued analysis and its applications. Nevertheless, there are some limitations for using this metric. For example, if we have a sequence $\{A_k\}_{k=1}^\infty$ of strictly convex compact sets and $h(A_k, A) \rightarrow 0$, then the limit set A needs not be strictly convex. Indeed, consider on the Euclidean plane the following ellipsoids

$$A_k = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + k^2 x_2^2 \leq 1\}.$$

Each set A_k is strictly convex, but the limit set $A = \{(x_1, 0) \in \mathbb{R}^2 \mid x_1 \in [-1, 1]\}$ is not strictly convex. Note, that strict convexity of the set means differentiability of the supporting function of this set. This fact is very useful for applications. Below we give some sufficient conditions for the limit of a sequence of strictly convex compacta to be also strictly convex.

We say that a sequence of convex compacta $\{A_k\}_{k=1}^\infty \subset \mathbb{R}^n$ is *uniformly convex with modulus* δ if $\inf_k \text{diam } A_k > 0$ and the function $\delta(\varepsilon)$, $\varepsilon \in [0, \inf_k \text{diam } A_k]$, is continuous and has the property $0 < \delta(\varepsilon) \leq \delta_{A_k}(\varepsilon)$ for all $\varepsilon \in (0, \inf_k \text{diam } A_k)$ and for all k .

Lemma 1.1. Let a sequence $\{A_k\}_{k=1}^\infty \subset \mathbb{R}^n$ of convex compacta converge to a convex compactum A in the Hausdorff metric. If the sequence $\{A_k\}_{k=1}^\infty$ is uniformly convex with modulus δ , $\delta : (0, \varepsilon_0] \rightarrow (0, +\infty)$, then the compactum A is a uniformly convex set with the modulus $\delta_A(\varepsilon) \geq \delta(\varepsilon)$, $0 < \varepsilon \leq \varepsilon_0$. In particular, this implies strict convexity of the set A .

Proof. Choose arbitrary points $x, y \in A$ with $\|x - y\| < \varepsilon_0$. There are two sequences $\{x_k\} \subset A_k$, $\{y_k\} \subset A_k$ such that $x_k \rightarrow x$, $y_k \rightarrow y$, $k \rightarrow \infty$. For all sufficiently large k we have $\|x_k - y_k\| < \varepsilon_0$. Due to the uniform convexity of the sequence $\{A_k\}$ we obtain that

$$\frac{x_k + y_k}{2} + B_{\delta(\|x_k - y_k\|)}(0) \subset A,$$

and

$$\left(p, \frac{x_k + y_k}{2}\right) + \delta(\|x_k - y_k\|)\|p\| \leq s(p, A_k), \quad \forall p \in \mathbb{R}^n.$$

Taking the limit $k \rightarrow \infty$, using (1.2) and the continuity of the function δ we get

$$\left(p, \frac{x + y}{2}\right) + \delta(\|x - y\|)\|p\| \leq s(p, A), \quad \forall p \in \mathbb{R}^n,$$

i.e.

$$s\left(p, \frac{x + y}{2} + B_{\delta(\|x - y\|)}(0)\right) \leq s(p, A), \quad \forall p \in \mathbb{R}^n.$$

By the Separation Theorem [9, 13] we obtain the following

$$\frac{x + y}{2} + B_{\delta(\|x - y\|)}(0) \subset A.$$

□

2. THE MAIN PROPERTIES OF METRIC ρ

In general, the subdifferential of a convex function is only upper semicontinuous [1, 13]. For (not strictly) convex compactum A the sets $A(p)$ are also upper semicontinuous with respect to p . This leads to the fact that in the formula (1.3) from Definition 1.3 one cannot replace \sup by \max .

Example 2.1. Consider in \mathbb{R}^3 two sets:

$$A = \text{co} \left\{ \{(x_1, x_2, x_3) \mid (x_1 - 1)^2 + x_2^2 = 1; x_3 = 0\} \cup \{(0, 0, 1)\} \right\},$$

$$B = \text{co} \left\{ \{(x_1, x_2, x_3) \mid (x_1 - 1)^2 + x_2^2 + x_3^2 = 1; x_3 = 0\} \cup \{(0, 0, 1)\} \right\}.$$

It is easy to see that $B \subset A$, $\text{diam } B = \text{diam } A = \sqrt{5}$, and $\text{diam } A$ and $\text{diam } B$ are attained only on the the line segment $[(0, 0, 1), (2, 0, 0)] \subset B$.

Let $a_k \in \{(x_1, x_2, x_3) \mid (x_1 - 1)^2 + x_2^2 = 1; x_2 < 0; x_3 = 0\}$ such that $a_k \rightarrow (2, 0, 0)$. The line segment $[(0, 0, 1), a_k]$ is a generatrix of the cone A for all k .

Let H_k be a supporting plane of the set A such that $[(0, 0, 1), a_k] \subset H_k$. Let p_k be a unit normal vector to the plane H_k such that $(p_k, a_k) > 0$. It is easy to see that $p_k \rightarrow p_0 = \frac{1}{\sqrt{5}}(1, 0, 2)$.

For any k we have $B(p_k) = \{(0, 0, 1)\}$ and $A(p_k) = H_k \cap A = [(0, 0, 1), a_k]$. By Definition 1.3 it follows that

$$\rho(A, B) \geq h(A(p_k), B(p_k)) = h(\{(0, 0, 1)\}, \{(0, 0, 1), a_k\}) = \|(0, 0, 1) - a_k\| = \sqrt{\|a_k\|^2 + 1},$$

and $\sqrt{\|a_k\|^2 + 1} \rightarrow \sqrt{5}$, $\sqrt{\|a_k\|^2 + 1} < \sqrt{5}$ for all k . However, the only line segment which realizes $\text{diam } A = \text{diam } B = \sqrt{5}$ is the line segment $[(0, 0, 1), (2, 0, 0)] \subset A \cap B$. Thus $\rho(A, B) = \lim_{k \rightarrow \infty} h(A(p_k), B(p_k)) = \sqrt{5}$, but for all p , $\|p\| = 1$, $h(A(p), B(p)) < \sqrt{5}$. □

Lemma 2.1. *Let $A \subset \mathbb{R}^n$ be a convex compactum. If the set-valued mapping $B_1(0) \ni p \rightarrow A(p)$ is lower semicontinuous, then the compactum A is strictly convex.*

Proof. Suppose that there exists $p \in B_1(0)$ such that the set $A(p)$ is not a singleton. Let $\{x, y\} \subset A(p)$, $x \neq y$, and $q = \frac{y - x}{\|y - x\|}$. Obviously, q is orthogonal to p .

Consider a sequence $\{q_k\}_{k=1}^\infty \subset \text{cone}\{p, q\}$ such that $q_k \rightarrow p$, $k \rightarrow \infty$, $\|q_k\| = 1$ and $q_k \neq p$ for all k . Let $H_p^- = \{z \in \mathbb{R}^n \mid (p, z) \leq s(p, A)\}$, $H_{q_k}^+ = \{z \in \mathbb{R}^n \mid (q_k, z) \geq (y, q_k)\}$, $H_q^+ = \{z \in \mathbb{R}^n \mid (q, z) \geq (y, q)\}$.

By lower semicontinuity of $A(\cdot)$ we have for any $\varepsilon > 0$ and for all sufficiently large k

$$A(p) \subset A(q_k) + B_\varepsilon(0). \quad (2.4)$$

On the other hand,

$$A(q_k) \subset H_{q_k}^+ \cap H_p^- \subset H_q^+ \cap H_p^-. \quad (2.5)$$

Due to the inclusion (2.5) we obtain that

$$\varrho(x, A(q_k)) \geq \varrho(x, H_q^+ \cap H_p^-) = \|x - y\| > 0. \quad (2.6)$$

Inequality (2.6) implies that for all k

$$x \notin A(q_k) + \frac{\|x - y\|}{2} B_1(0).$$

This contradicts the inclusion (2.4). \square

Lemma 2.2. *Consider a sequence $F_k : (T, \varrho) \rightarrow 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ of set-valued mappings which are upper (lower) semicontinuous with compact images. Let the sequence $\{F_k(t)\}_{k=1}^{\infty}$ uniformly converge to the set-valued mapping $F : (T, \varrho) \rightarrow 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ in the Hausdorff metric, i.e.*

$$\forall \varepsilon > 0 \exists k_\varepsilon \forall k > k_\varepsilon \forall t \in T \quad h(F_k(t), F(t)) < \varepsilon.$$

Then the set-valued mapping F is upper (lower) semicontinuous on T .

Proof. The proof is a standard argument of uniform convergence. \square

We shall write $F_k \rightrightarrows F$, $t \in T$, in the case of uniform convergence on the set T of the sequence F_k to the mapping F .

Theorem 2.1. *The metric space of convex compacta in \mathbb{R}^n is complete with respect to metric ρ .*

Proof. Let $\{A_k\}_{k=1}^{\infty}$ be a fundamental sequence of convex compacta with respect to metric ρ . This means that

$$\forall \varepsilon > 0 \exists M \forall k, m > M \forall p \in B_1(0) \quad h(A_m(p), A_k(p)) < \varepsilon.$$

By convexity of compact sets $A_m(p)$ and completeness of the space of convex compacta with respect to the Hausdorff metric (see [9, Theorem 1.3.2]) we obtain that $A_m(p) \rightrightarrows A_p$, $p \in B_1(0)$, and the set A_p is convex and compact for all $p \in \mathbb{R}^n$, $\|p\| = 1$.

Put

$$A = \text{cl co} \bigcup_{\|p\|=1} A_p.$$

For any $q \in B_1(0)$ and any $x(q) \in A_q$ there exists a sequence $\{x_m(q)\}_{m=1}^{\infty}$ such that $x_m(q) \in A_m(q)$ for all m and $x_m(q) \rightarrow x(q)$. Taking a limit $m \rightarrow \infty$ in the inequality $(p, x_m(p)) \geq (p, x_m(q))$, we obtain $(p, x(p)) \geq (p, x(q))$. Hence $(p, x(p)) \geq s(p, A)$ and $x(p) \in A(p)$, i.e. $A_p \subset A(p)$.

The converse inclusion $A(p) \subset A_p$ can be proved on the contrary with the help of separation theorem. \square

Corollary 2.1. *The metric subspace of strictly convex compacta in \mathbb{R}^n is complete with respect to metric ρ .*

Proof. The proof is analogous to the proof of Theorem 2.1 except that all sets $A_m(p)$, A_p are singletons. \square

Suppose that A, B are convex compacta. By formula (1.2) we have

$$\rho(A, B) = \sup_{\|p\|=1} \sup_{\|q\|=1} |s(q, A(p)) - s(q, B(p))|,$$

and hence

$$\rho(A, B) \geq \sup_{\|p\|=1} |s(p, A(p)) - s(p, B(p))| = \sup_{\|p\|=1} |s(p, A) - s(p, B)| = h(A, B). \quad (2.7)$$

Thus $\rho(A_k, A) \rightarrow 0$ implies that $h(A_k, A) \rightarrow 0$.

Theorem 2.2. *The metric space of strictly convex compacta in \mathbb{R}^n is not locally compact with respect to the metric ρ .*

Proof. Choose a sequence $\{A_k\}_{k=1}^{\infty}$ of strictly convex compacta such that $A_k \subset B_R(0)$ for all k and there exists a nonstrictly convex compactum A with $h(A_k, A) \rightarrow 0$.

Suppose that a subsequence $\{A_{k_m}\}_{m=1}^{\infty}$ converges to a compactum B in the metric ρ .

From the estimate $\rho(A_{k_m}, B) \geq h(A_{k_m}, B)$ and $h(A_{k_m}, A) \rightarrow 0$ we get equality $B = A$. So $\rho(A_{k_m}, A) \rightarrow 0$. This means that $A_{k_m}(p) \rightrightarrows A(p)$, $\|p\| = 1$. But the set $A_{k_m}(p)$ is a singleton for

all m and p . By the choice of A there exists $p_0 \in \mathbb{B}_1(0)$ such that the set $A(p_0)$ is not a singleton. Contradiction. \square

Further we shall obtain the estimate of distance $\rho(A, B)$ via $h(A, B)$ for some convex closed sets in a Banach space. In a Banach space E for closed convex bounded sets $A, B \subset E$ we define

$$\rho(A, B) = \sup_{\|p\|_* = 1} h(A(p), B(p)).$$

If the space E is reflexive then $A(p) \neq \emptyset$, $B(p) \neq \emptyset$ for all $p \in E^*$.

Note that if the space E contains a uniformly convex nonsingleton set then such space E has equivalent uniformly convex norm [3, Theorem 2.3]. In particular, such space E is reflexive.

Note also that for any uniformly convex set A we have that $\text{diam } A < +\infty$ and the modulus of convexity $\delta_A(\varepsilon)$ is a strictly increasing function when $\varepsilon \in [0, \text{diam } A)$ [3].

Theorem 2.3. *Let E be a Banach space. Let $A, B \subset E$ be convex closed bounded sets. Let the set A be a nonsingleton and a uniformly convex set with the modulus of convexity δ_A . Let $\Delta = \lim_{t \rightarrow \text{diam } A - 0} \delta_A(t)$. Then*

$$\rho(A, B) \leq \begin{cases} h(A, B) + \delta_A^{-1}(h(A, B)), & h(A, B) < \Delta, \\ h(A, B) \left(1 + \frac{\text{diam } A}{\Delta}\right), & h(A, B) \geq \Delta, \end{cases} \quad (2.8)$$

where the function δ_A^{-1} is the inverse function to the function δ_A . Furthermore, if the set A is a singleton then $\rho(A, B) = h(A, B)$.

Proof. Let $h = h(A, B)$. Suppose that A is not a singleton. Fix $p \in \mathbb{B}_1^*(0)$. Let $A(p) = \{a(p)\}$. Fix an arbitrary point $b(p) \in B(p)$.

Case 1. $h < \Delta$. Choose $t > 1$ such that $th < \Delta$.

Subcase 1.1. $s(p, A) \geq s(p, B)$. By formula (1.2) we have $0 \leq (p, a(p)) - (p, b(p)) \leq h$. Let $a \in A$ be such a point that $a \in b(p) + B_{th}(0)$.

Define $H_A(p) = \{z \in E \mid (p, z) = s(p, A)\}$, $H_A^-(p) = \{z \in E \mid (p, z) \leq s(p, A)\}$, $H_B(p) = \{z \in E \mid (p, z) = s(p, B)\}$.

We have $\varrho(b(p), H_A(p)) = (p, a(p) - b(p)) \leq h$, $\varrho(a, H_A(p)) \leq \|a - b(p)\| + \varrho(b(p), H_A(p)) \leq (1+t)h$ and $A \cup B \subset H_A^-(p)$. Hence the line segment $[a(p), a]$ belongs to the set $H_A(p)^-$. Let $w = \frac{a(p)+a}{2}$, $\varrho(w, H_A(p)) = \frac{1}{2}\varrho(a, H_A(p)) \leq \frac{1+t}{2}h$. By the inclusion

$$w + \delta_A(\|a(p) - a\|)B_1(0) \subset A \subset H_A^-(p)$$

we get

$$\delta_A(\|a(p) - a\|) \leq \varrho(w, H_A(p)) \leq \frac{1+t}{2}h.$$

Hence $\|a(p) - a\| \leq \delta_A^{-1}\left(\frac{1+t}{2}h\right)$. Thus we obtain that

$$\|a(p) - b(p)\| \leq \|a(p) - a\| + \|a - b(p)\| \leq \delta_A^{-1}\left(\frac{1+t}{2}h\right) + th,$$

i.e. $b(p) \in a(p) + \left(\delta_A^{-1}\left(\frac{1+t}{2}h\right) + th\right)B_1(0)$. Due to the arbitrary choice of the point $b(p) \in B(p)$ we have

$$B(p) \subset a(p) + \left(\delta_A^{-1}\left(\frac{1+t}{2}h\right) + th\right)B_1(0)$$

and

$$h(A(p), B(p)) = h(\{a(p)\}, B(p)) \leq \delta_A^{-1}\left(\frac{1+t}{2}h\right) + th.$$

Taking the limit $t \rightarrow 1 + 0$, we obtain that

$$h(A(p), B(p)) = h(\{a(p)\}, B(p)) \leq \delta_A^{-1}(h) + h.$$

Subcase 1.2. $s(p, A) < s(p, B)$. Then all arguments of the subcase 1.1 still apply except that

$$\varrho(a, H_A(p)) \leq \varrho(a, H_B(p)) \leq \|a - b(p)\| \leq th,$$

$\varrho(w, H_A(p)) \leq \frac{t}{2}h$, $\|a(p) - a\| \leq \delta_A^{-1}(\frac{t}{2}h)$. Hence

$$h(A(p), B(p)) = h(\{a(p)\}, B(p)) \leq \delta_A^{-1}\left(\frac{t}{2}h\right) + th.$$

Taking the limit $t \rightarrow 1 + 0$, we obtain that

$$h(A(p), B(p)) \leq \delta_A^{-1}\left(\frac{1}{2}h\right) + h.$$

So again when $h < \Delta$ we have for all $p \in \mathbb{B}_1^*(0)$

$$h(A(p), B(p)) \leq \delta_A^{-1}(h) + h.$$

Hence $\rho(A, B) = \sup_{\|p\|=1} h(A(p), B(p)) \leq \delta_A^{-1}(h) + h$.

Case 2. $h \geq \Delta$. Then for any $t > 1$ we have

$$\rho(A, B) \leq \text{diam } A + th \leq \frac{h}{\Delta} \text{diam } A + th \leq h \left(t + \frac{\text{diam } A}{\Delta} \right), \quad \forall t > 1.$$

Taking the limit $t \rightarrow 1 + 0$, we get

$$\rho(A, B) \leq h \left(1 + \frac{\text{diam } A}{\Delta} \right).$$

In the case when A is a singleton the equality $\rho(A, B) = h(A, B)$ follows by definition 1.3. \square

For a set $A \subset \mathbb{R}^n$, $A \subset B_R(a)$ for some $a \in \mathbb{R}^n$ and $R > 0$, we define *R-strongly convex hull* of the set A , as the intersection of all closed balls of radius R each of which contains the set A . We shall denote the R -strongly convex hull of the set A by $\text{strco}_R A$ (cf. [2]).

Example 2.2. The estimate (2.8) is exact. Consider two sets A and B on the Euclidean plane \mathbb{R}^2 . Let $0 < \varepsilon < R$, $a \left(\sqrt{2R\varepsilon - \varepsilon^2}, 0 \right) \in \mathbb{R}^2$ and

$$A = \text{strco}_R \{ B_\varepsilon((0, 0)) \cup \{a\} \} + B_R(0), \quad B = \text{strco}_R \{ B_\varepsilon(a) \cup \{(0, 0)\} \} + B_R(0).$$

Let $p = (0, 1)$. It is easy to see that $h(A, B) = \varepsilon$, $A(p) = (\varepsilon + R)p$, $B(p) = \left(\sqrt{2R\varepsilon - \varepsilon^2}, \varepsilon \right) + Rp$. Hence

$$\rho(A, B) \geq h(A(p), B(p)) = \|a\| = \sqrt{2R\varepsilon - \varepsilon^2} = \sqrt{2Rh(A, B) - h^2(A, B)}.$$

The sets A and $B = a - A$ are intersections of closed balls of radius $R + \varepsilon$ and $\delta_A(s) = \delta_B(s) \geq (R + \varepsilon)\delta_{\mathcal{H}}\left(\frac{s}{R + \varepsilon}\right)$, where $\delta_{\mathcal{H}}(s) = 1 - \sqrt{1 - \frac{s^2}{4}}$ is the modulus of convexity for the Hilbert space (see [7, p. 63]). Thus $\delta_A(s) = \delta_B(s) \geq \frac{s^2}{8(R + \varepsilon)}$, and $\delta_A^{-1}(t) \leq 2\sqrt{2(R + \varepsilon)t}$. So the order of $h(A, B)$ in formula (2.8) is exact. \square

Remark 2.1. The result of Theorem 2.3 was proved for p -convex sets in [8, Formula (5)]. Note that any p -convex set, $p > 0$, in the paper [8] is in fact the intersection of closed balls of radius $R = \frac{1}{2p}$. From the definition of p -convex set (inequality (2) of [8]) it follows that for any p -convex set $A \subset \mathbb{R}^n$, any point $a \in A$ and any unit vector $w \in \{q \in \mathbb{R}^n \mid (q, x - a) \leq 0, \forall x \in A\}$ we have

$$(w, x - a) + p\|x - a\|^2 \leq 0, \quad \forall x \in A,$$

or

$$A \subset B_R(a - Rw), \quad \text{where } R = \frac{1}{2p}.$$

Hence $A = \bigcap_{\|w\|=1} B_{\frac{1}{2p}}\left(a(w) - \frac{1}{2p}w\right)$, where $\{a(w)\} = A(w)$.

This also follows by results of [5], [9, Chapter 3].

Corollary 2.2. *Suppose that $F_i : (T, \varrho) \rightarrow 2^{\mathbb{R}^n} \setminus \{\emptyset\}$, $i = 1, 2$, are continuous (in the metric ρ) set-valued mappings with strictly convex images. Let $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear operator. Then the set-valued mappings $F_1(t) + F_2(t)$, $LF_1(t)$, $F_2(t) \stackrel{*}{-} F_1(t) = \bigcap_{x \in F_1(t)} (F_2(t) - x)$, $F_1(t) \cap F_2(t)$ (the*

latter two if nonempty) are continuous in the metric ρ .

Proof. The proof is similar for all cases. Let us prove the continuity of $F_1(t) \cap F_2(t)$.

The continuity of set-valued mappings F_i in the metric ρ and formula (2.7) gives the continuity of set-valued mappings F_i in the Hausdorff metric.

It is well known that the intersection of two continuous in the Hausdorff metric set-valued mappings with compact strictly convex images is also continuous in the Hausdorff metric (cf. [1, 9]). Thus the set-valued mapping $H = F_1 \cap F_2 : (T, \varrho) \rightarrow 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ is continuous in the Hausdorff metric.

For any point $t = t_0 \in T$ the set $H(t_0)$ is a strictly(=uniformly) convex compactum from \mathbb{R}^n with some modulus of convexity δ_{t_0} . By Theorem 2.3 we have $\rho(H(t), H(t_0)) \leq$

$$\leq \max \left\{ h(H(t), H(t_0)) + \delta_{t_0}^{-1}(h(H(t), H(t_0))); \left(1 + \frac{\text{diam } H(t_0)}{\Delta} \right) h(H(t), H(t_0)) \right\} \xrightarrow{t \rightarrow t_0} 0,$$

where $\Delta = \delta_{t_0}(\text{diam } H(t_0))$. If $H(t_0)$ is a singleton then $\rho(H(t), H(t_0)) = h(H(t), H(t_0)) \xrightarrow{t \rightarrow t_0} 0$. \square

3. APPLICATIONS

3.1. We prove a theorem about smooth approximation of the extremal problem.

Theorem 3.1. *Let $F : (T, \varrho) \rightarrow 2^{\mathbb{R}^n} \setminus \emptyset$ be a continuous set-valued mapping with compact convex images and suppose that there exists $r > 0$ such that for all $t \in T$ $F(t) \subset B_r(a(t))$ for some $a(t) \in \mathbb{R}^n$. Let $\text{diam } F(t) \geq d > 0$ for all $t \in T$.*

For any $t \in T$ and $p \in \mathbb{R}^n$, $\|p\| = 1$, consider the following problem

$$\max\{(p, x) \mid x \in F(t)\}. \quad (3.9)$$

Then for any $\varepsilon \in (0, 1)$ there exists an approximation $F_\varepsilon : (T, \varrho) \rightarrow 2^{\mathbb{R}^n} \setminus \emptyset$, $F(t) \subset F_\varepsilon(t)$ for all $t \in T$, $h(F(t), F_\varepsilon(t)) \leq \varepsilon$ for all $t \in T$, such that for each $t \in T$ and $p \in \mathbb{R}^n$, $\|p\| = 1$, the following problem

$$\max\{(p, x) \mid x \in F_\varepsilon(t)\} \quad (3.10)$$

has a unique solution $F_\varepsilon(t, p) = \{f_\varepsilon(t, p)\} = \arg \max_{x \in F_\varepsilon(t)} (p, x)$ which is Hölder continuous with the power $\frac{1}{2}$ with respect to $h(F(t_1), F(t_2))$ for all $t_1, t_2 \in T$. The power $\frac{1}{2}$ is the best possible in the general case.

Proof. Fix $\varepsilon \in (0, 1)$. Let $R = \max\{\frac{r^2}{\varepsilon}, r + 1\}$. Define $F_\varepsilon(t)$ as the intersection of all closed balls of radius R , each of which contains the set $F(t)$. This set is nonempty because $F(t) \subset B_R(a(t))$.

By [2, formulae (5.7), (5.8)] and [9, Theorem 4.4.7] we have for all $t_1, t_2 \in T$

$$h(F_\varepsilon(t_1), F_\varepsilon(t_2)) \leq C(\varepsilon)h(F(t_1), F(t_2)), \quad C(\varepsilon) = \max \left\{ \sqrt{\frac{R+r}{R-r}}, 1 + \frac{r^2}{R(R-r)} \right\}.$$

By [2, Theorem 5.4] and [9, Theorem 4.4.6] we have

$$h(F(t), F_\varepsilon(t)) \leq \frac{r^2}{R} \leq \varepsilon, \quad \forall t \in T.$$

By the inequality $\delta_{F_\varepsilon(t)}(s) \geq R\delta_{\mathcal{H}}\left(\frac{s}{R}\right)$, where $\delta_{\mathcal{H}}(s) = 1 - \sqrt{1 - \frac{s^2}{4}}$ is the modulus of convexity for the Hilbert space [7, p. 63], we get

$$\delta_{F_\varepsilon(t)}(s) \geq \frac{s^2}{8R}, \quad \forall s \in (0, \text{diam } F_\varepsilon(t)),$$

and by Theorem 2.3 we obtain for any $p \in B_1(0)$ that

$$\begin{aligned} \|f_\varepsilon(t_1, p) - f_\varepsilon(t_2, p)\| &\leq \\ &\leq \max\{C(\varepsilon)h(F(t_1), F(t_2)) + \sqrt{8RC(\varepsilon)h(F(t_1), F(t_2))}; (1 + \frac{2r}{\Delta})h(F(t_1), F(t_2))\}, \end{aligned}$$

where $\Delta = R\delta_{\mathcal{H}}(\frac{d}{R})$. On the other hand, we have for any convex compact set $A \subset \mathbb{R}^n$ that for some constant $C > 0$ the inequality $\delta_A(\varepsilon) \leq C\varepsilon^2$ holds for all $\varepsilon \in (0, \text{diam } A)$ (see [3]). Taking into account also Example 2.2, we see that the power $\frac{1}{2}$ is the best possible. \square

3.2. We consider Lipschitz selections and parametrizations of (strictly) convex compact sets with metric ρ .

With any convex compact set $A \subset \mathbb{R}^n$ we can associate the *Steiner point*

$$s(A) = \frac{1}{v_1} \int_{\|p\|=1} s(p, A)p d\mu_{n-1}, \quad v_1 = \mu_n B_1(0),$$

where μ_n is the Lebesgue measure in \mathbb{R}^n .

It is well known that the Steiner point is a Lipschitz selection of convex compacta in \mathbb{R}^n with the Hausdorff metric, i.e. for any convex compacta $A, B \subset \mathbb{R}^n$ we have $s(A) \in A$ and

$$\|s(A) - s(B)\| \leq \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{n}{2} + 1)}{\Gamma(\frac{n+1}{2})} h(A, B).$$

The Lipschitz constant (of the order \sqrt{n}) above is the best possible [11]. See also [14, P. 53], [12], [9, Theorem 2.1.2] for details.

Using the Gauss-type formula (see [9, formula (2.1.15)], [12, formula (3.1)]) we obtain that

$$\frac{1}{v_1} \int_{\|p\|=1} s(p, A)p d\mu_{n-1} = \frac{1}{v_1} \int_{\|p\|\leq 1} \nabla s(p, A) d\mu_n.$$

Note that $\nabla s(p, A)$ exists a.e. on the ball $B_1(0)$.

For any convex compactum $A \subset \mathbb{R}^n$ define $U(A) = \{p \in B_1(0) \mid \exists \nabla s(p, A)\}$. The function $s(p, A)$ is Lipschitz continuous hence $\mu_n U(A) = \mu_n B_1(0)$. Let $a(A, p) = \nabla s(p, A)$ for $p \in U(A)$ and $a(A, p) = 0$ for $p \in B_1(0) \setminus U(A)$.

Let $A, B \subset \mathbb{R}^n$ be convex compacta and $U = U(A) \cap U(B)$, $\mu_n U = \mu_n B_1(0)$. Then

$$\|s(A) - s(B)\| \leq \frac{1}{v_1} \int_U \|a(A, p) - a(B, p)\| d\mu_n \leq \frac{1}{v_1} \int_U \rho(A, B) d\mu_n = \rho(A, B).$$

Thus the Steiner point is a Lipschitz selection of convex compacta in \mathbb{R}^n with metric ρ with the Lipschitz constant 1.

Let \mathbb{A} be a collection of strictly convex compacta. Then for any $p \in \mathbb{R}^n$, $\|p\| = 1$, the function $a(p) = A(p)$, $A \in \mathbb{A}$, is a Lipschitz selection of the family \mathbb{A} with the Lipschitz constant 1 in the metric ρ .

Theorem 3.2. *Let a collection of strictly convex compacta \mathbb{A} from \mathbb{R}^n be uniformly bounded, i.e. there exists $M > 0$ such that $\|A\| = h(\{0\}, A) \leq M$ for all $A \in \mathbb{A}$.*

Then there exists the family of functions

$$f_{\lambda, p} : \mathbb{A} \rightarrow \mathbb{R}^n, \quad (\lambda, p) \in [0, 1] \times B_1(0), \quad (3.11)$$

such that for any $A \in \mathbb{A}$ we have

$$A = \{f_{\lambda, p}(A) \mid \lambda \in [0, 1], p \in B_1(0)\}$$

and for any $(\lambda, p) \in [0, 1] \times B_1(0)$ the function $f_{\lambda, p}$ is Lipschitz on $A \in \mathbb{A}$ selection in the metric ρ with Lipschitz constant 1.

Moreover, the function $[0, 1] \times B_1(0) \ni (\lambda, p) \rightarrow f_{\lambda, p}(A)$ is continuous for any $A \in \mathbb{A}$ and the function $f_{\lambda, p}$ is additive: $f_{\lambda, p}(A + B) = f_{\lambda, p}(A) + f_{\lambda, p}(B)$, $A, B \in \mathbb{A}$.

PROOF. For any $A \in \mathbb{A}$ we define $f_{\lambda,p}(A) = \lambda a(p) + (1 - \lambda)s(A) \in A$. Let $A, B \in \mathbb{A}$. Then (note, that $b(p) = B(p)$ for any $p \in \mathbb{B}_1(0)$)

$$\begin{aligned} \|f_{\lambda,p}(A) - f_{\lambda,p}(B)\| &\leq \lambda \|a(p) - b(p)\| + (1 - \lambda) \|s(A) - s(B)\| \leq \\ &\leq \lambda \rho(A, B) + (1 - \lambda) \rho(A, B) = \rho(A, B). \end{aligned}$$

Choose $\lambda_1, \lambda_2 \in [0, 1]$ and $p_1, p_2 \in \mathbb{B}_1(0)$ and $A \in \mathbb{A}$.

$$\begin{aligned} \|f_{\lambda_1,p_1}(A) - f_{\lambda_2,p_2}(A)\| &= \|\lambda_1 a(p_1) + (1 - \lambda_1)s(A) - \lambda_2 a(p_2) - (1 - \lambda_2)s(A)\| \leq \\ &\leq \|\lambda_1 a(p_1) - \lambda_2 a(p_2)\| + |\lambda_1 - \lambda_2| \|s(A)\| \leq |\lambda_1 - \lambda_2| \|a(p_1)\| + |\lambda_2| \|a(p_1) - a(p_2)\| + \\ &\quad + |\lambda_1 - \lambda_2| \|s(A)\| \leq 2|\lambda_1 - \lambda_2|M + \|a(p_1) - a(p_2)\|. \end{aligned}$$

The gradient $a(p) = \nabla s(p, A)$ for the strictly convex compact set A is uniformly continuous on the unit sphere (see [3, Lemma 2.2]). So the function $[0, 1] \times \mathbb{B}_1(0) \ni (\lambda, p) \rightarrow f_{\lambda,p}(A)$ is uniformly continuous.

By the Moreau-Rockafellar theorem [13] for all $A, B \in \mathbb{A}$ we get $A(p) + B(p) = (A + B)(p)$ for all $p \in \mathbb{B}_1(0)$. Using the additive property of the Steiner point [9], [12], [14] we obtain that $f_{\lambda,p}$ is an additive selection for all $\lambda \in [0, 1]$ and $\|p\| = 1$. \square

REMARK 3.1. We see from the proof of Theorem 3.2, that the function

$$(\lambda, p) \rightarrow f_{\lambda,p}(A)$$

is uniformly continuous for any $A \in \mathbb{A}$. More precisely, $f_{\lambda,p}$ is Lipschitz on $\lambda \in [0, 1]$ (with Lipschitz constant $2M$) and uniformly continuous on $p \in \mathbb{B}_1(0)$. Note that $f_{\lambda,p}(A)$ is Lipschitz on $p \in \mathbb{B}_1(0)$ if and only if the set A is an intersection of closed balls of the same fixed radius. The last assertion follows by results of [5] and by Theorem 4.3.2 of [9]: *a set A is the intersection of closed balls of fixed radius $R > 0$ in Hilbert space if and only if $\|a(p) - a(q)\| \leq R\|p - q\|$ for all $p, q \in \mathbb{B}_1(0)$.* Here $a(p) = A(p)$.

ACKNOWLEDGEMENTS

This research was supported by SRA grants P1-0292-0101, J1-2057-0101, and BI-RU/10-11/002. The first author was supported by RFBR grant 10-01-00139-a, ADAP project "Development of scientific potential of higher school" 2.1.1/11133 and projects of FAP "Kadry" 1.2.1 grant P938 and grant 16.740.11.0128. We thank the referee for comments and suggestions.

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